Math 3236 Statistical Theory

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$\underline{\theta} \in \Omega$

$f(x \mid \underline{\theta})$ dist on x given $\underline{\theta}$

$\xi(\underline{\theta})$ dist on $\underline{\theta}$ prior

$f(\underline{x} \mid \underline{\theta}) = \prod_{i=1}^{n} f(x_i \mid \underline{\theta})$

j. dist of a random sample

$L(\underline{\theta}) = f(\underline{x} \mid \underline{\theta})$

likelihood function.

Updating rule

prior is $\xi(\theta)$

posterior dist. on θ given x,

$$\xi(\theta \mid x_1) = \frac{f(x_1 \mid \theta)\,\xi(\theta)}{\int f(x_1 \mid \theta)\,\xi(\theta)\,d\theta}$$

$\int f(x_1 \mid \theta)\,\xi(\theta)\,d\theta$ is the probability of observing x.

x_2

$$\xi(\theta \mid x_1, x_2) =$$

$$\frac{f(x_2 \mid \theta)\,\xi(\theta \mid x_1)}{\int f(x_2 \mid \theta)\,\xi(\theta \mid x_1)\,d\theta}$$

$$\int f(x_2 \mid \theta)\,\xi(\theta \mid x_1)\,d\theta = f(x_2 \mid x_1)$$

$$f(x_2 \mid \theta)\,\xi(\theta \mid x_1) =$$

$$f(x_2 \mid \theta)\,\frac{f(x_1 \mid \theta)\,\xi(\theta)}{f(x_1)}$$

$$\int f(x_2 | \theta)\, \xi(\theta | x_1)\, d\theta =$$

$$\frac{\int f(x_2 | \theta)\, f(x_1 | \theta)\, \xi(\theta)\, d\theta}{f(x_1)}$$

$$\xi(\theta | x_1, x_2) = \frac{f(x_1 | \theta)\, f(x_2 | \theta)\, \xi(\theta)}{\int f(x_1 | \theta)\, f(x_2 | \theta)\, \xi(\theta)\, d\theta}$$

Assume That

X_i are Bernoolli

$p(1 | \theta) = \theta \qquad \theta \in [0, 1]$

$p(0 | \theta) = 1 - \theta$

$\xi(\theta)$ is Beta with par α, β

If I observe N values

of which n are 1 Then

The posterior is $B(\alpha + n, \beta + N - n)$

The family of Beta distr. is conjugated to the Bernoulli.

If my prior is $B(1,1)$
[uniform]

$$\mathbb{E}(\Xi \mid \underline{x}) = \int \theta \, \xi(\theta \mid \underline{x}) \, d\theta$$

$$= \frac{n(\underline{x}) + 1}{N + 2}$$

$n(\underline{x})$ = number of 1 in $\underline{x}$

prior is α, β

$$\mathbb{E}(\Xi \mid \underline{x}) = \frac{\alpha + n(\underline{x})}{N + \alpha + \beta}$$

If I flip a coin N Times and I see n ones, I would estimate

$$\hat{\theta} = \frac{n}{N}$$

This is what I get if I assume The prior To be

B $\alpha = \beta = 0$.

$$B(\alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$B(0, 0)$ is called an <u>improper</u> <u>prior</u>.

α, β are called

hyperparameters.

Prior is uniform $\alpha = \beta = 1$

n ones m zeros

$B(n+1, m+1)$

$$V(n,m) = \frac{(n+1)(m+1)}{(n+m+2)^2(n+m+3)}$$

I want $V \leq 0.01$

$$\max_{n+m=N} V(n,m)$$

$$V(n,n) = \frac{(n+1)^2}{4(n+1)^2(2n+3)}$$

$$V(n,m) \leq \frac{1}{4(N+3)}$$

$N \geqslant 22$!

Sensibility Analysis.

$$\hat{\theta} = \frac{\alpha + n(\underline{x})}{\alpha + \beta + N}$$

N number of observations

$n(\underline{x})$ number of 1 in my obs.

Conjugated Families.

Poisson:

X_i have a Poisson dist

par θ , $i = 1 \dots N$

X is discrete

$$\mathbb{P}(X = x) = \frac{\theta^x}{x!} e^{-\theta}$$

$$\Omega = \mathbb{R}^+$$

$\xi(\theta)$ is Gamma dist

with hyp. α, β

$$\alpha \to \alpha + \sum_i x_i$$

$$\beta \to \beta + N$$

If I start $\alpha = 1$ $\beta = 2$

N observation $\quad y = \sum_i x_i$

$$V = \frac{y+1}{(N+2)^2}$$

Normal distribution.

$X_i \quad i = 1 \ldots N$

$N(\theta, \sigma^2) \quad \sigma$ is known

The prior $\xi(\theta)$ is

$$N(\mu_0, v_0^2)$$

posterior is (if $\bar{x} = \frac{1}{N}\sum_i x_i$)

$$N(\mu_1, v_1^2)$$

$$\mu_1 = \frac{\sigma^2 \mu_0 + N \bar{x} v_0^2}{\sigma^2 + N v_0^2}$$

$$v_1^2 = \frac{\sigma^2 v_0^2}{\sigma^2 + N v_0^2}$$

$$X_i \simeq N(\theta, \sigma^2)$$

$$\xi(\theta) \simeq N(\mu_0, v_0^2)$$

$$\xi(\theta \mid \bar{x}) = N(\mu_1, v_1^2)$$